Proof theory of predicate logic

Natural deduction rules

Proofs in the natural deduction calculus for predicate logic are similar to those for propositional logic in except that we have new proof rules for dealing with the quantifiers and with the equality symbol. Strictly speaking, we are overloading the previously established proof rules for the propositional connectives ∧, ∨ etc. That simply means that any proof rule of still valid for logical formulas of predicate logic (we originally defined those rules for logical formulas of propositional logic). As in the natural deduction calculus for propositional logic, the additional rules for the quantifiers and equality will come in two flavors: introduction and elimination rules.

The proof rules for equality First, let us state the proof rules for equality. Here equality does not mean syntactic, or intensional, equality, but equality in terms of computation results. In either of these senses, any term t has to be equal to itself. This is expressed by the introduction rule for equality:

$$\frac{}{t = t} =\mathrm{i}$$

which is an axiom (as it does not depend on any premises). Notice that it may be invoked only if t is a term, our language doesn't permit us to talk about equality between formulas. This rule is quite evidently sound, but it is not very useful on its own. What we need is a principle that allows us to substitute equals for equals repeatedly. For example, suppose that y ∗ (w + 2) equals y ∗ w + y ∗ 2; then it certainly must be the case that z ≥ y ∗ (w + 2) implies z ≥ y ∗ w + y ∗ 2 and vice versa. We may now express this substitution principle as the rule =e:

$$\frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} =\mathrm{e}.$$

Note that t1 and t2 have to be free for x in φ, whenever we want to apply the rule =e; this is an example of a side condition of a proof rule.